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Current forms and gauge invariance

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Abstract
Let C be the bundle of connections of a principal G-bundle $\pi: P \to M$, and let $\mathcal{V}$ be the vector bundle associated with P by a linear representation $G \to GL(V)$ on a finite-dimensional vector space V. The Lagrangians on $J^1(C \times_M \mathcal{V})$ whose current form is gauge invariant, are described and the gauge-invariant Lagrangians on $J^1(\mathcal{V})$ are classified.

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1. Introduction

Let $\pi: P \to M$ be a principal bundle with structure group a Lie group G, let $G \to GL(V)$ be a linear representation on a finite-dimensional real vector space and let $\pi_{\mathcal{V}}: \mathcal{V} = (P \times V)/G \to M$ be the associated vector bundle. Also, let $p: C = C(P) \to M$ be the bundle of connections of the given principal bundle.

Lagrangian functions on the interaction bundle $\bar{\pi}: C \times_M \mathcal{V} \to M$ which are invariant under the gauge group of P, can be geometrically characterized by means of the generalized curvature map

$$\begin{cases} \Omega: J^1(C \times_M \mathcal{V}) \to \mathcal{K} = \mathcal{V} \oplus (T^*M \otimes \mathcal{V}) \oplus \left(\bigwedge^2 T^*M \otimes \operatorname{ad} P\right) \\ \Omega\left(j^1_x\sigma, j^1_x\xi\right) = \left(\xi(x), (\nabla^\sigma \xi)_x, \Omega^\sigma_x\right) \end{cases} \tag{1}$$

where ∇^σ stands for the covariant derivative induced by the connection Γ^σ on $\mathcal{V}$, and Ω^σ is the curvature of Γ^σ, considered as a 2-form on M with values in the adjoint bundle. Then, a Lagrangian $L\colon J^1(C\times_M\mathcal{V})\to\mathbb{R}$ is gauge invariant if and only if L factors through Ω by means of a gauge-invariant function,

$$\hat{L}\colon\mathcal{V}\oplus(T^*M\otimes\mathcal{V})\oplus\left(\bigwedge^2 T^*M\otimes\operatorname{ad}P\right)\to\mathbb{R}$$

such that $L=\hat{L}\circ\Omega$ (see [1]).

If a Lagrangian is gauge invariant, then its associated current form also is; but the converse does not hold: there exist non-invariant Lagrangians having a gauge-invariant current form.

The current form that we have just mentioned does not refer to the Noether current attached to an infinitesimal symmetry in the classical field theory; rather, it is a 'universal' form defined on the 1-jet bundle of the interacting bundle, which is associated with a given Lagrangian density. Such a form originates from the following difficulty appearing in the minimal coupling setting.

Let Θ_{L_0} be the Poincaré–Cartan n-form ($n=\dim M$) attached to a Lagrangian density $\Lambda_0\colon J^1\mathcal{V}\to\bigwedge^n T^*M$, $\Lambda_0=L\mathbf{v}$. For a gauge vector field X on P, we can consider the Noether current $(n-1)$-form $i_{X_\mathcal{V}^{(1)}}\Theta_{L_0}$, where $X_\mathcal{V}^{(1)}$ is the 1-jet prolongation of the natural lift of X to $\mathcal{V}$ (see its definition in section 2.5). Then, the assignment $X\mapsto i_{X_\mathcal{V}^{(1)}}\Theta_{L_0}$ is $C^\infty(M)$-linear. When Λ_0 is not gauge invariant (which is the usual case in field theories, where Λ_0 is just G-invariant) the current form does not provide a conservation law. One can fix this issue by considering gauge 'potentials'. More precisely, by means of the so-called Utiyama trick (cf [1, 16]), one defines a Lagrangian $\Lambda\colon J^1(C\times_M\mathcal{V})\to\bigwedge^n T^*M$ associated with Λ_0 by coupling the matter field with gauge potentials. In this case, the Noether current is obtained as $i_{\bar{X}^{(1)}}\Theta_L$, where $\bar{X}=(X_C,X_\mathcal{V})$ is the natural lift of X to the interaction bundle, X_C being the natural lift of X to C (see the definitions in section 2.3). The problem is that the assignment $X\mapsto i_{\bar{X}^{(1)}}\Theta_L$ is no longer $C^\infty(M)$-linear with respect to $X_C^{(1)}$. The way to overcome this difficulty is to make the interior product with the second component of the vector field only; that is, to consider the assignment $X\mapsto i_{X_\mathcal{V}^{(1)}}\Theta_L$. Then, the new assignment becomes linear, as proved in section 3, and this property enables us to see the previous map as an $(\operatorname{ad}P)^*$-valued $(n-1)$-form J_L on $J^1(C\times_M\mathcal{V})$ (see the formula (8)), which is called the universal current form attached to Λ.

The form J_L is used in writing the Noether conservation law attached to a gauge symmetry and also in formulating the inhomogeneous field equations for minimally coupled Lagrangian densities (see [2] and section 3). Moreover, the universal current forms of some special Lagrangians also play a basic role in computing gauge invariants of topological nature; e.g., see [4].

The goal of this paper is to obtain the characterization—in terms of the geometry of the interaction bundle—of the Lagrangians having a gauge-invariant universal current form. In fact, while gauge invariance of the Lagrangian function is a reasonable requirement from the point of view of physics, it is also interesting to look for the characterization of those Lagrangians whose current form is gauge invariant, as the latter object has the advantage of being directly observable. The main result is theorem 5.2 which states that a Lagrangian $L\colon J^1(C\times_M\mathcal{V})\to\mathbb{R}$ defines a gauge-invariant current form J_L if and only if L can be written as $L=L'+L''$, where L' is a gauge-invariant Lagrangian and L'' is a Lagrangian factoring through the first partial derivatives of a Hilbert–Nagata basis for the algebra of G-invariant polynomials (see the formula (22) below) on a dense open subset. In theorem 4.4 we previously obtained the classification of those gauge-invariant Lagrangians depending only on the matter field.

The present paper may be considered as a strong generalization to arbitrary structure groups of the results given in [6] for Abelian groups, in the setting of classical electromagnetism.

2. Notation and preliminaries

2.1. Jet bundles

Throughout this paper, Greek indices run from 1 to m, Latin indices other than a, b, c, d run from 1 to n, and the indices a, b, c, d run from 1 to r.

Given a fibre bundle $\pi\colon E \to M$, we denote by $\pi_{10}\colon J^1E \to E$ the 1-jet bundle of local sections of π, which is an affine bundle modelled over the vector bundle $\pi^*T^*M \otimes VE$, where $VE \subset TE$ is the vector sub-bundle of π-vertical tangent vectors. If (x^i, y^α) is a fibred coordinate system for π defined on an open subset $V \subseteq E$, we denote by $\left(x^i, y^\alpha; y^\alpha_i\right)$ the coordinate system induced on $\pi_{10}^{-1}(V)$, that is, $y^\alpha_i\left(j^1_x s\right) = (\partial(y^\alpha \circ s)/\partial x^i)(x)$.

If $\Phi\colon E \to E$ is a bundle morphism whose projection $\varphi\colon M \to M$ is a diffeomorphism, then we define its 1-jet prolongation $\Phi^{(1)}\colon J^1E \to J^1E$ as

$$\Phi^{(1)}\left(j^1_x s\right) = j^1_{\varphi(x)}(\Phi \circ s \circ \varphi^{-1}).$$

Accordingly, we denote by $X^{(1)} \in \mathfrak{X}(J^1E)$ the infinitesimal generator of the flow $\Phi^{(1)}_t$, Φ_t being the flow of a π-projectable vector field $X \in \mathfrak{X}(E)$.

2.2. Principal bundles

A gauge transformation of a principal G-bundle $\pi\colon P \to M$ is a diffeomorphism $\Phi\colon P \to P$ such that $\pi \circ \Phi = \pi$ and $R_g \circ \Phi = \Phi \circ R_g$, $\forall\, g \in G$, where R_g stands for the right action of g on P. The set Gau P of all gauge transformations of P is a group under composition. An infinitesimal gauge transformation is a π-vertical vector field $X \in \mathfrak{X}(P)$ such that $R_g \cdot X = X$, $\forall\, g \in G$; i.e., X is G-invariant. It is readily seen that X is an infinitesimal gauge transformation if and only if its flow is a one-parameter subgroup of Gau P. Because of this, we denote by gau P the algebra of all infinitesimal gauge transformations. Let $\operatorname{ad} P = (P \times \mathfrak{g})/G$ denote the adjoint bundle; i.e., the bundle associated with P by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Then, there is a one-to-one correspondence between the algebra of sections of this bundle and the gauge algebra; that is, $\Gamma(M, \operatorname{ad} P) \simeq \operatorname{gau} P$. Indeed, a section ζ of ad P can be seen as the G-invariant, π-vertical vector field X on P defined as follows. If $\zeta(x)$ equals the coset $(u, B)_G \in \operatorname{ad} P$ of the pair $(u, B) \in P \times \mathfrak{g}$, $u \in \pi^{-1}(x)$, then X along the fibre through u is defined by $X_{u\cdot g} = (\operatorname{Ad}_{g^{-1}} B)^*_{u\cdot g}$, $g \in G$, where the star superscript denotes the infinitesimal generator of the G-action; i.e., A^*_u is the tangent vector at $t = 0$ to the curve $t \mapsto u \cdot \exp(tA)$, for every $A \in \mathfrak{g}$.

2.3. Principal connections

The group G acts naturally on TP and the quotient $(TP)/G$ is a fibre bundle over M. A connection on $\pi\colon P \to M$ can be seen as a splitting of the exact sequence $0 \to \operatorname{ad} P \to (TP)/G \to TM \to 0$. We thus define the bundle of connections $\pi_C\colon C \to M$ to be the bundle whose fibre over $x \in M$ is $C_x = \{\lambda\colon T_xM \to ((TP)/G)_x \mid \pi_* \circ \lambda = \mathrm{id}\}$. In this way, we obtain an affine bundle modelled over the vector bundle $T^*M \otimes \operatorname{ad} P \to M$, whose global sections $\sigma\colon M \to C$ correspond to principal connections Γ_σ on $P \to M$ (e.g., see [2, 5, 7, 9, 10, 12]). We have $\dim C = n + nm$, where $n = \dim M$, $m = \dim G$.

If $\Phi\colon P \to P$ is a gauge transformation, then $\Phi_*\colon TP \to TP$ satisfies the condition $(R_g)_* \circ \Phi_* = \Phi_* \circ (R_g)_*$, and we can thus project it onto the quotient $(\Phi_*)_G\colon (TP)/G \to (TP)/G$. We define a bundle morphism $\Phi_C\colon C \to C$ as $\Phi_C(\lambda) = (\Phi_*)_G \circ \lambda$, $\lambda \in C$, which is, in fact, a diffeomorphism. Given a section $\sigma\colon M \to C$ with corresponding connection Γ_σ, it is readily checked that the connection defined by the section $\Phi_C \circ \sigma$ is no other than the image

$\Phi(\Gamma^\sigma)$ of Γ^σ by Φ according to the standard theory of connections (cf [13, II. proposition 6.1]). The map Gau $P \to$ Diff C, $\Phi \mapsto \Phi_C$, is a group morphism. If Φ_t is the flow of an element $X \in$ gau P and X_C is the infinitesimal generator of the flow $(\Phi_t)_C$, then we have a Lie algebra morphism gau $P \to \mathfrak{X}(C)$, $X \mapsto X_C$.

2.4. Coordinates on C

Let (U, x^i) be an open coordinate domain in M such that $\pi^{-1}(U) \cong U \times G$, and let (B_α) be a basis of the Lie algebra $\mathfrak{g}$. We obtain a coordinate system $\left(x^i, A_j^\alpha\right)$ on $\pi_C^{-1}(U)$ by setting $\lambda(\partial/\partial x^j) = \partial/\partial x^j + A_j^\alpha(\lambda)\tilde{B}_\alpha$, $\forall\,\lambda \in \pi_C^{-1}(U)$, where $\tilde{B}$ is the infinitesimal generator of the gauge flow $\Phi_t^B(x, g) = (x, \exp(tB)\cdot g)$, $B \in \mathfrak{g}$. Note that $\tilde{B}$ is a G-invariant vector field and hence it can be seen as a section of ad $P \hookrightarrow (TP)/G$. In fact, $(\tilde{B}_\alpha \bmod G)$ is a basis of the $C^\infty(U)$-module $\Gamma(U, \mathrm{ad}\,P)$. Every infinitesimal gauge transformation $X \in$ gau P can be expressed on $\pi^{-1}(U)$ as

$$X = g^\alpha \tilde{B}_\alpha \qquad g^\alpha \in C^\infty(U). \tag{2}$$

Hence, we have

$$X_C = -\left(\frac{\partial g^\alpha}{\partial x^j} - c_{\beta\gamma}^\alpha g^\beta A_j^\gamma\right)\frac{\partial}{\partial A_j^\alpha} \tag{3}$$

where $c_{\beta\gamma}^\alpha$ are the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis (B_α).

2.5. Coordinates on $\mathcal{V}$

Let $G \to GL(V) \cong GL(r, \mathbb{R})$, $r = \dim V$, be a linear representation on a real vector space and let $\pi_{\mathcal{V}}: \mathcal{V} = (P \times V)/G \to M$ be the associated vector bundle. Let $(u, v)_G \in (P \times V)/G$ denote the coset of the pair $(u, v) \in P \times V$ modulo G. Every $\Phi \in$ Gau P induces a vector bundle morphism $\Phi_{\mathcal{V}}: \mathcal{V} \to \mathcal{V}$, by setting $\Phi_{\mathcal{V}}((u, v)_G) = (\Phi(u), v)_G$. The map Gau $P \to$ Diff $\mathcal{V}$, $\Phi \mapsto \Phi_{\mathcal{V}}$, is a group morphism that induces a Lie algebra morphism gau $P \to \mathfrak{X}(\mathcal{V})$, $X \mapsto X_{\mathcal{V}}$. We also have an identification $\pi_{\mathcal{V}}^{-1}(U) \cong U \times V$ given by $((x, g), v)_G \mapsto (x, g\cdot v)$, $x \in U$, $g \in G$, $v \in V$. Therefore, given a basis (v_a) of V, we obtain a natural coordinate system (x^i, y^a) on $\pi_{\mathcal{V}}^{-1}(U)$, by setting $v = y^a(v)v_a$, $v \in \pi_{\mathcal{V}}^{-1}(U)$, and we have

$$X_{\mathcal{V}} = -g^\alpha (\dot{B}_\alpha)_b^a y^b \frac{\partial}{\partial y^a} \tag{4}$$

where $(\dot{B})_b^a$ stands for the matrix of $B \in \mathfrak{g}$ under the Lie algebra representation $\mathfrak{g} \to \mathfrak{gl}(r, \mathbb{R})$ defined by the action $G \to Gl(r, \mathbb{R})$, with respect to the basis (v_a).

2.6. Invariance on the interaction bundle

Every $\Phi \in$ Gau P induces a bundle diffeomorphism $\bar{\Phi}: C \times_M \mathcal{V} \to C \times_M \mathcal{V}$ on the interaction bundle $\bar{\pi}: C \times_M \mathcal{V} \to M$, defined as $\bar{\Phi} = (\Phi_C, \Phi_{\mathcal{V}})$. Furthermore, every $X \in$ gau P defines the vector field $\bar{X} = (X_C, X_{\mathcal{V}}) = X_C + X_{\mathcal{V}}$, which is tangent to the submanifold $C \times_M \mathcal{V} \subset C \times \mathcal{V}$; i.e., $\bar{X} \in \mathfrak{X}(C \times_M \mathcal{V})$.

A Lagrangian $L: J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ is said to be gauge invariant if

$$\bar{X}^{(1)}(L) = 0 \qquad \forall\, X \in \text{gau}\, P. \tag{5}$$

Similarly, a Lagrangian $L: J^1C \to \mathbb{R}$ (respectively $L: J^1\mathcal{V} \to \mathbb{R}$) is gauge invariant if $X_C^{(1)}(L) = 0$ (respectively $X_{\mathcal{V}}^{(1)}(L) = 0$), for every $X \in$ gau P.

Remark 2.1. We can define the gauge invariance of a Lagrangian L on $C \times_M \mathcal{V}$ (respectively C and $\mathcal{V}$) by setting $L \circ \bar{\Phi}^{(1)} = L$ (respectively $L \circ \Phi_C^{(1)} = L$ and $L \circ \Phi_{\mathcal{V}}^{(1)} = L$) for every $\Phi \in \mathrm{Gau}\, P$, but the definition of invariance under infinitesimal gauge transformations is more useful for practical purposes. Anyway, if the group G is connected (which is the case for most field theories) both notions coincide.

According to the standard formulae for jet prolongation, the formulae (3) and (4) yield

$$X_C^{(1)} = -\left(\frac{\partial g^\alpha}{\partial x^j} - c^\alpha_{\beta\gamma} g^\beta A^\gamma_j\right)\frac{\partial}{\partial A^\alpha_j} - \left(\frac{\partial^2 g^\alpha}{\partial x^i \partial x^j} - c^\alpha_{\beta\gamma}\frac{\partial g^\beta}{\partial x^i} A^\gamma_j - c^\alpha_{\beta\gamma} g^\beta A^\gamma_{j,i}\right)\frac{\partial}{\partial A^\alpha_{j,i}} \tag{6}$$

$$X_{\mathcal{V}}^{(1)} = -g^\alpha (\dot{B}_\alpha)^a_b y^b \frac{\partial}{\partial y^a} - \left(\frac{\partial g^\alpha}{\partial x^i}(\dot{B}_\alpha)^a_b y^b + g^\alpha (\dot{B}_\alpha)^a_b y^b_i\right)\frac{\partial}{\partial y^a_i}. \tag{7}$$

We thus obtain the local expression for the invariance condition (5).

3. Gauge invariance of the current form

Let $\Lambda \colon J^1(C \times_M \mathcal{V}) \to \bigwedge^n T^*M$ be a Lagrangian density. Let us assume that M is connected and oriented by a fixed volume form $\mathbf{v} \in \Omega^n(M)$. Hence we can write $\Lambda = L\mathbf{v}$, with $L \in C^\infty(J^1(C \times_M \mathcal{V}))$. Let Θ_L be the Poincaré–Cartan form associated with Λ, which is an n-form on $J^1(C \times_M \mathcal{V})$, $(n-1)$-horizontal with respect to the projection $\bar{\pi}_1 \colon J^1(C \times_M \mathcal{V}) \to M$, induced from $\bar{\pi} \colon C \times_M \mathcal{V} \to M$.

The map

$$\begin{aligned} \mathrm{gau}\, P &\to \bar{\pi}_1^* \Omega^{n-1}(M) \\ X &\mapsto i_{X_{\mathcal{V}}^{(1)}} \Theta_L \end{aligned}$$

is $C^\infty(M)$-linear. Indeed, from the local expression of the Poincaré–Cartan form (e.g., see [8]) in this case we obtain

$$\Theta_L = (-1)^{i+1}\frac{\partial L}{\partial A^\alpha_{j,i}}\left(\mathrm{d}A^\alpha_j - A^\alpha_{j,k}\,\mathrm{d}x^k\right)\wedge \mathbf{v}_i + (-1)^{i+1}\frac{\partial L}{\partial y^c_i}\left(\mathrm{d}y^c - y^c_k\,\mathrm{d}x^k\right)\wedge \mathbf{v}_i + L\mathbf{v}$$

where the coordinates (x^i) on M are assumed to be adapted to $\mathbf{v}$; i.e.,

$$\begin{aligned} \mathbf{v} &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ \mathbf{v}_i &= \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n. \end{aligned}$$

Hence, by using the formulae (2) and (7), we obtain

$$i_{X_{\mathcal{V}}^{(1)}} \Theta_L = (-1)^i \frac{\partial L}{\partial y^c_i} g^\alpha (\dot{B}_\alpha)^c_d y^d \mathbf{v}_i$$

where we have used the same notation as in section 2.4. This equation exhibits the $C^\infty(M)$-linearity of the assignment.

Moreover, from the same equation it follows that the form $i_{X_{\mathcal{V}}^{(1)}} \Theta_L$ at a point $\left(j^1_x\sigma, j^1_x\xi\right)$ in $J^1(C \times_M \mathcal{V})$ depends only on the value $X(x)$ with respect to its argument $X \in \mathrm{gau}\, P$. Taking account of the fact that the space of sections of $\mathrm{ad}\, P$ can be identified to the gauge algebra, we can thus define an $(\mathrm{ad}\, P)^*$-valued $(n-1)$-form J on $J^1(C \times_M \mathcal{V})$ by setting

$$(J_L)_{(j^1_x\sigma, j^1_x\xi)}(X_x) = \left(i_{X_{\mathcal{V}}^{(1)}}\Theta_L\right)_{(j^1_x\sigma, j^1_x\xi)} \qquad X_x \in (\mathrm{ad}\, P)_x, \quad x \in M \tag{8}$$

where $X \in \mathrm{gau}\, P$ is any section which takes the value X_x at $x \in M$. Locally,

$$J_L = (-1)^i \frac{\partial L}{\partial y^c_i} (\dot{B}_\alpha)^c_d y^d \mathbf{v}_i \otimes \tilde{B}^\alpha \tag{9}$$

where $(\tilde{B}^\alpha)$ stands for the basis of sections of $(\mathrm{ad}\,P)^*$ dual to $(\tilde{B}_\alpha)$. The form J_L is called the *universal current form* associated with Λ (cf [2, 11]).

Such a form plays an important role in field theories. The Noether theorem can be formulated in terms of the universal current as follows: if $L\colon J^1(C\times_M\mathcal{V})\to\mathbb{R}$ is a gauge-invariant Lagrangian and $(\sigma,\xi)\colon M\to C\times_M\mathcal{V}$ is an extremal of $L\mathbf{v}$, then the Noether invariant corresponding to $X\in\mathrm{gau}\,P$ is given by

$$i_{X_C^{(1)}}\Theta_L + J_L(X) \tag{10}$$

and the Noether conservation law simply states that the form obtained by pulling (10) back along $(j^1\sigma, j^1\xi)$ is closed.

The current form also appears in the Euler–Lagrange equations of the so-called minimally coupled Lagrangians. Actually, the complete description of these systems is given by a Lagrangian $L_C\colon J^1C\to\mathbb{R}$ on the space of connections, such as, for example, the Yang–Mills Lagrangian, and an interaction Lagrangian $L_I\colon C\times_M J^1\mathcal{V}\to\mathbb{R}$. Hence we are led to consider the variational problem defined by the sum $L\colon J^1(C\times_M\mathcal{V})\to\mathbb{R}$, $L=L_C+L_I$. The Euler–Lagrange equations of Lagrangians of this type are written as follows. First of all, we introduce a definition. Since $J^1C\to C$ is an affine bundle modelled over the bundle $\pi_C^*T^*M\otimes VC$, given a section σ of $C\to M$, the vertical differential of the Lagrangian L_C along $j^1\sigma$ gives rise to a section of the bundle $\pi_C^*TM\otimes V^*C$. We can identify VC to $T^*M\otimes\mathrm{ad}\,P$ along σ and we thus obtain a section of the bundle $TM\otimes TM\otimes(\mathrm{ad}\,P)^*$, which is denoted by Ξ_{L_C}. Then, a section $(\sigma,\xi)\colon M\to C\times_M\mathcal{V}$ is critical for L if and only if the following equation holds,

$$\mathrm{d}^\sigma(\Xi_{L_C}\lrcorner\mathbf{v}) = (j^1\phi)^*J_{L_I}$$

where d^σ is the covariant exterior differential defined by the connection σ on $(\mathrm{ad}\,P)^*$-valued forms on M, and $\lrcorner$ denotes the contraction of the two contravariant components of the tensor Ξ_{L_C} with the volume form $\mathbf{v}$. For a proof of this result we refer the reader to [11].

Usually (see [12, section 37]), a current form is defined to be a differential $(n-1)$-form on M taking values in the coadjoint bundle. The current form, as defined above, induces a current form in the usual sense for every extremal of $\Lambda=L\mathbf{v}$ by simply pulling J_L back along the 1-jet prolongation of the extremal.

Now, we would like to define the Lie derivative of the current form. The standard Lie derivative does not make sense for this, as the current form takes values in a vector bundle. We remark on the fact that we only need to define the Lie derivative with respect to gauge vector fields. For such fields, the definition can be stated as follows.

Let J be a $(\mathrm{ad}\,P)^*$-valued form on $J^1(C\times_M\mathcal{V})$. For every $X\in\mathrm{gau}\,P$, we define the Lie derivative $\mathrm{L}_{\tilde{X}^{(1)}}J$ to be the only $(\mathrm{ad}\,P)^*$-valued form satisfying

$$\langle\mathrm{L}_{\tilde{X}^{(1)}}J, Y\rangle = \mathrm{L}_{\tilde{X}^{(1)}}\langle J, Y\rangle - \langle J,[X,Y]\rangle \tag{11}$$

for every $Y\in\mathrm{gau}\,P=\Gamma(\mathrm{ad}\,P)$, where $\langle,\rangle$ denotes the pairing between $\mathrm{ad}\,P$ and $(\mathrm{ad}\,P)^*$ induced by duality, and the Lie derivative on the right-hand side is the standard one. For the local expression of this operator, if $X=g^\alpha\tilde{B}_\alpha$ is written as in (2), we decompose $J=J_\alpha\otimes\tilde{B}^\alpha$, J_α being scalar forms on $J^1(C\times_M\mathcal{V})$, and we obtain

$$\mathrm{L}_{\tilde{X}^{(1)}}J = \left(\mathrm{L}_{\tilde{X}^{(1)}}J_\alpha + J_\beta g^\gamma c^\beta_{\gamma\alpha}\right)\otimes\tilde{B}^\alpha. \tag{12}$$

Proposition 3.1 (infinitesimal functoriality of the universal current form)**.** *Let $L\colon J^1(C\times_M\mathcal{V})\to\mathbb{R}$ be a Lagrangian and J_L its current form. Then*

$$\mathrm{L}_{\tilde{X}^{(1)}}J_L = J_{\tilde{X}^{(1)}(L)}. \tag{13}$$

Proof. From the definition of J_L and $\mathrm{L}_{\bar{X}^{(1)}}$ given in the formulae (8) and (11) respectively, we obtain

$$\begin{aligned}\langle \mathrm{L}_{\bar{X}^{(1)}} J, Y\rangle &= \mathrm{L}_{\bar{X}^{(1)}} \langle J, Y\rangle + \langle J, [X, Y]\rangle \\ &= \mathrm{L}_{\bar{X}^{(1)}}\big(i_{Y^{(1)}_{\mathcal{V}}}\Theta_L\big) - i_{[X,Y]^{(1)}_{\mathcal{V}}}\Theta_L \\ &= i_{Y^{(1)}_{\mathcal{V}}}(\mathrm{L}_{\bar{X}^{(1)}}\Theta_L) + i_{[\bar{X}^{(1)}, Y^{(1)}_{\mathcal{V}}]}\Theta_L - i_{[X,Y]^{(1)}_{\mathcal{V}}} \\ &= i_{Y^{(1)}_{\mathcal{V}}}(\mathrm{L}_{\bar{X}^{(1)}}\Theta_L)\end{aligned}$$

where, in the last step, we have used the identities

$$\begin{aligned}[X, Y]^{(1)}_{\mathcal{V}} &= \big[X^{(1)}_{\mathcal{V}}, Y^{(1)}_{\mathcal{V}}\big] \\ &= \big[X^{(1)}_{\mathcal{V}} + X^{(1)}_{C}, Y^{(1)}_{\mathcal{V}}\big] \\ &= \big[\bar{X}^{(1)}, Y^{(1)}_{\mathcal{V}}\big].\end{aligned}$$

The proof is complete by recalling the infinitesimal functoriality of the Poincaré–Cartan form; that is, $\mathrm{L}_{\bar{X}^{(1)}}\Theta_L = \Theta_{\bar{X}^{(1)}(L)}$ (cf [8], proposition 2.2]). □

For a given Lagrangian L, the form J_L is said to be gauge invariant if $\mathrm{L}_{\bar{X}^{(1)}} J_L = 0$, for every $X \in \mathrm{gau}\, P$. From proposition 3.1 and the formula (9), the local expression of the gauge invariance of the current form reads

$$\frac{\partial(\bar{X}^{(1)}(L))}{\partial y^a_i}(\dot{B})^a_b y^b = 0 \qquad \forall B \in \mathfrak{g}, \forall X \in \mathrm{gau}\, P. \tag{14}$$

Corollary 3.2. *If $L: J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ is a gauge-invariant Lagrangian, the current form is gauge invariant as well.*

4. Gauge invariance in $\mathcal{V}$

4.1. Invariant polynomials

Here we use the same notation as in section 2.5. A polynomial ρ on V is said to be G-invariant if $\rho(g \cdot v) = \rho(v), \forall v \in V, g \in G$. Let I^G_d be the space of G-invariant homogeneous polynomials of degree d and let $I^G = \oplus_d I^G_d$ be the $\mathbb{Z}$-graded algebra of all G-invariant polynomials. Each $\rho \in I^G$ induces a differentiable function $\rho_{\mathcal{V}}: \mathcal{V} \to \mathbb{R}$ sending $\rho_{\mathcal{V}}((u, v)_G) = \rho(v)$. The definition makes sense as ρ is G-invariant.

If $\rho_1, \ldots, \rho_k \in I^G$, then we denote by $\mathcal{P}_{\mathcal{V}}: \mathcal{V} \to \mathbb{R}^k$ the map whose components are $(\rho_1)_{\mathcal{V}}, \ldots, (\rho_k)_{\mathcal{V}}$. We rather think of $\mathcal{P}_{\mathcal{V}}$ as being a map of fibred manifolds over M, thus writing $\mathcal{P}_{\mathcal{V}}: \mathcal{V} \to M \times \mathbb{R}^k$ instead of $(\pi_V, \mathcal{P}_{\mathcal{V}})$.

We now define the map

$$\begin{cases}\bar{\mathcal{P}}_{\mathcal{V}}: J^1\mathcal{V} \to \pi^*_{\mathcal{V}}(\oplus^k T^*M) \\ \bar{\mathcal{P}}_{\mathcal{V}}\big(j^1_x\xi\big) = (\xi(x), (\mathrm{d}(\rho^1 \circ \xi)_x, \ldots, \mathrm{d}(\rho^k \circ \xi)_x))\end{cases} \tag{15}$$

for any local section ξ of $\pi_{\mathcal{V}}: \mathcal{V} \to M$. In local coordinates, for a single $\rho \in I^G$ the expression of $\bar{\mathcal{P}}_{\mathcal{V}}$ is given by

$$\bar{\mathcal{P}}_{\mathcal{V}}\big(x^i, y^a, y^a_i\big) = \left(x^i, y^a, \frac{\partial\rho}{\partial y^a} y^a_j\, \mathrm{d}x^j\right). \tag{16}$$

Proposition 4.1. *For any polynomials $\rho^1, \ldots, \rho^k \in I^G$, the map (15) is gauge equivariant; that is, $\bar{\mathcal{P}}_{\mathcal{V}} \circ \Phi^{(1)}_{\mathcal{V}} = \Phi_{\mathcal{V}} \circ \bar{\mathcal{P}}_{\mathcal{V}}$, for all $\Phi \in$ Gau P.*

(Note that, as $\Phi: P \to P$ projects onto the identity map on M, the action of $\Phi_{\mathcal{V}}$ along the fibres of $\pi_{\mathcal{V}}^*(\oplus^k T^*M)$ is trivial.)

Proof. We can assume P is trivial: $P = M \times G$; hence $\mathcal{V} = M \times V$. Accordingly, every gauge transformation $\Phi: P \to P$ can be written as $\Phi(x, g) = (x, \phi(x)g)$ where $\phi \in C^\infty(M, G)$, and similarly, every section $\xi: M \to \mathcal{V}$ can be written as $\xi(x) = ((x, e); v(x))_G$ where $v \in C^\infty(M, V)$. Then, we have

$$\begin{aligned}\left(\bar{\mathcal{P}}_{\mathcal{V}} \circ \Phi_{\mathcal{V}}^{(1)}\right)\left(j_{x_0}^1 \xi\right) &= (\Phi_{\mathcal{V}}(\xi(x)), (\mathrm{d}(\rho^1 \circ \phi \cdot v)_x, \ldots, \mathrm{d}(\rho^k \circ \phi \cdot v)_x)) \\ &= (\Phi_{\mathcal{V}}(\xi(x)), (\mathrm{d}(\rho^1 \circ v)_x, \ldots, \mathrm{d}(\rho^k \circ v)_x)) \\ &= (\Phi_{\mathcal{V}} \circ \bar{\mathcal{P}}_{\mathcal{V}})\left(j_x^1 \xi\right)\end{aligned}$$

thus concluding. □

Corollary 4.2. *For every invariant polynomial* ρ*, the following equations hold:*

$$(\dot{B}_\alpha)^c_d y^d \frac{\partial \rho}{\partial y^c} = 0 \qquad 1 \leqslant \alpha \leqslant m$$

$$(\dot{B}_\alpha)^c_d y^d \frac{\partial}{\partial y^c}\left(\frac{\partial \rho}{\partial y^a} y_i^a\right) + (\dot{B}_\alpha)^c_d y_j^d \frac{\partial}{\partial y_j^c}\left(\frac{\partial \rho}{\partial y^a} y_i^a\right) = 0 \qquad 1 \leqslant \alpha \leqslant m \quad 1 \leqslant i \leqslant n.$$

Proof. From the formulae (7) and (16) the result follows. □

4.2. The distributions $\mathcal{D}'$ and $\mathcal{D}''$

For the study of the gauge invariance of a current form, first we need to characterize the solutions to the following system of mn equations,

$$(\dot{B})^a_b y^b \frac{\partial f}{\partial y_i^a} = 0 \qquad \forall B \in \mathfrak{g} \quad \forall i \tag{17}$$

for $f \in C^\infty(J^1\mathcal{V})$ or $f \in C^\infty(C \times_M \mathcal{V})$. By using the formula (7), it is readily checked that the solutions to (17) are the first integrals of the distribution $\mathcal{D}'$ on $J^1\mathcal{V}$ defined by

$$\mathcal{D}'_{j_x^1\xi} = \left\{\left(X_{\mathcal{V}}^{(1)}\right)_{j_x^1\xi} : X \in \operatorname{gau} P, X_x = 0\right\} \qquad j_x^1\xi \in J^1\mathcal{V}. \tag{18}$$

Proposition 4.3. *Assume that G is compact and that I^G is generated by the polynomials $\rho_1, \ldots, \rho_k$. Then, there exists a dense open subset $O \subseteq J^1\mathcal{V}$ such that $f \in C^\infty(J^1\mathcal{V})$ is a first integral of $\mathcal{D}'$ if and only if it factors through*

$$\bar{\mathcal{P}}_{\mathcal{V}}: O \to \pi_{\mathcal{V}}^*(\oplus^k T^*M).$$

That is, $f = \bar{f} \circ \bar{\mathcal{P}}_{\mathcal{V}}$, where $\bar{f}: \pi_{\mathcal{V}}^*(\oplus^k T^*M) \to \mathbb{R}$ is an arbitrary differentiable function.

Proof. Given the action of a Lie group G on a finite-dimensional vector space V, the dimension of the G-orbits is constant on a dense open subset $O_1 \subseteq V$, and on the complement of O_1, the G-orbits are of smaller dimension (e.g., see [3, theorem 3.1]). Moreover, it is clear that for every $\rho \in I^G$ and every $v_0 \in V$, $y_0^a = y^a(v_0)$, we have $T_{v_0}(G \cdot v_0) \subseteq \ker(\mathrm{d}\rho)_{v_0}$. Hence,

$$\left\{(\dot{B})^c_d y_0^d \frac{\partial}{\partial y^c}\right\}_{B \in \mathfrak{g}} = T_{v_0}(G \cdot v_0) \subseteq \bigcap_{s=1}^k \ker(\mathrm{d}\rho^s)_{v_0}.$$

The dimension of $\bigcap_{s=1}^{k} \ker(\mathrm{d}\rho^s)_{v_0}$ is minimal on a dense open subset O_2 in V. If $\dim T_{v_0}(G \cdot v_0) < \dim \bigcap_{s=1}^{k} \ker(\mathrm{d}\rho^s)_{v_0}$ for every $v_0 \in O_1 \cap O_2$, we can construct (for example, by means of an auxiliary scalar product) a vector field X on V such that

$$X_v \notin T_v(G \cdot v) \qquad X_v \in \bigcap_{s=1}^{k} \ker(\mathrm{d}\rho^s)_v \qquad \forall\, v \in O_1 \cap O_2.$$

If $\alpha(t)$ is an integral curve of X, it is clear that $\mathrm{d}(\rho^s \circ \alpha)/\mathrm{d}t = 0$; hence $(\rho^s \circ \alpha)(t)$ is constant for every s. As X is transversal to G-orbits on $O_1 \cap O_2$, we have that the system of polynomials $\{\rho^s\}_{s=1}^{k}$ does not distinguish orbits, which is not true for compact groups (cf [3, 14]). Therefore, on $O = O_1 \cap O_2$, we have

$$\left\{ (\dot{B})^c_d y^d_0 \left(\frac{\partial}{\partial y^c} \right)_{v_0} \right\}_{B \in \mathfrak{g}} = T_{v_0}(G \cdot v_0) = \bigcap_{s=1}^{k} \ker(\mathrm{d}\rho^s)_{v_0}. \tag{19}$$

Hence the implicit equations defining the vector space $\left\langle (\dot{B})^c_d y^d_0 (\partial/\partial y^c)_{v_0} \right\rangle$ are

$$\frac{\partial \rho^s}{\partial y^c}(v_0) y^c = 0 \qquad 1 \leqslant s \leqslant k, \forall\, v_0 \in O \tag{20}$$

where we have used the natural identification $T_{v_0}(V) \cong V$.

Assume $\mathcal{V} = M \times V$; hence we have a natural identification

$$J^1(\mathcal{V}) = J^1(M, V) \cong (T^*M \otimes V) \oplus (M \times V).$$

Once a point $\xi_0 = (x_0, v_0) \in \mathcal{V}$, $v_0 \in O$ is fixed, the distribution $\mathcal{D}'$ restricts to a distribution $\mathcal{D}'_{\xi_0}$ along the fibre $(\pi_{\mathcal{V}})^{-1}_{10}(\xi_0) \cong T^*_{x_0} M \otimes V$, as $\mathcal{D}'$ is tangent to the fibres of $(\pi_{\mathcal{V}})_{10}$. For each index i, let V_i be the subspace of $(\pi_{\mathcal{V}})^{-1}_{10}(\xi_0)$ spanned by the vectors $(\mathrm{d}x^i)_{x_0} \otimes v_1, \ldots, (\mathrm{d}x^i)_{x_0} \otimes v_r$ and let y^a_i be the corresponding coordinates in V_i.

If f is a first integral of $\mathcal{D}'$ then, according to (17), for every $B \in \mathfrak{g}$, every index i, and every $v_0 \in O$, we have $(\dot{B})^a_b y^b_0 \big(\partial f/\partial y^a_i\big)\big(j^1_{x_0}\xi\big) = 0$, for all $j^1_{x_0}\xi$ in $(\pi_{\mathcal{V}})^{-1}_{10}(\xi_0)$. Hence $f|_{V_i}$ is a first integral of the distribution with constant coefficients spanned by the vector fields $(\dot{B})^c_d y^d_0 \partial\big/\partial y^c_i$, whose implicit equations are $(\partial\rho^s/\partial y^c)(v_0) y^c_i = 0$, $1 \leqslant s \leqslant k$, as follows from the formula (20).

Moreover, if $E_1 = c_1, \ldots, E_t = c_t$ are the equations defining the leaves of a distribution with constant coefficients, then the first integrals of such a distribution are $F(E_1, \ldots, E_t)$, where F is an arbitrary differentiable function in $\mathbb{R}^t$. Hence, for every $(x_0, v_0) \in M \times O \subset \mathcal{V}$ and every index i, the dependence of f with respect to the variables y^c_i is through the functions $(\partial\rho^s/\partial y^c)(v_0) y^c_i$. The proof is complete by taking into account the expression of $\bar{\mathcal{P}}_{\mathcal{V}}$ in the formula (16). □

Theorem 4.4. *With the same hypotheses and assumptions as in proposition 4.3, a Lagrangian* $L\colon J^1\mathcal{V} \to \mathbb{R}$ *is gauge invariant if and only if it factors through* $\bar{\mathcal{P}}_{\mathcal{V}}\colon O \to \pi^*_{\mathcal{V}}(\oplus^k T^*M)$.

Proof. From the expression of the vector field $X^{(1)}_{\mathcal{V}}$ given in the formula (7), we deduce that L is gauge invariant if and only if the following equations hold true:

$$(\dot{B}_\alpha)^a_b y^b \frac{\partial L}{\partial y^a} + (\dot{B}_\alpha)^a_b y^b_i \frac{\partial L}{\partial y^a_i} = 0 \qquad \forall\, \alpha$$

$$(\dot{B}_\alpha)^a_b y^b \frac{\partial L}{\partial y^a_i} = 0 \qquad \forall\, \alpha, i.$$

The second set of equations shows that L is a first integral of the distribution $\mathcal{D}'$. From proposition 4.3, we thus have $L = \tilde{L} \circ \bar{\mathcal{P}}_{\mathcal{V}}$; that is, $L\big(x^i, y^a, y^a_i\big) = \tilde{L}\big(x^i, y^a, (\partial\rho^s/\partial y^a) y^a_i\big)$

on the dense open subset $O \subset J^1\mathcal{V}$ for certain function $\tilde{L}$. By using the chain rule and taking corollary 4.2 into account, the first set of equations above yields

$$(\dot{B}_\alpha)^a_b y^b \frac{\partial \tilde{L}}{\partial y^a} = 0 \qquad \forall \alpha$$

which represents the invariance of $\tilde{L}$ with respect to the action of G on V. Then (see [14]) $\tilde{L}$ is a smooth function of the polynomials ρ^s with respect to the variables y^a, that is, we can write

$$\tilde{L}\left(x^i, y^a, \frac{\partial \rho^s}{\partial y^a} y^a_i\right) = \bar{L}\left(x^i, \rho^s, \frac{\partial \rho^s}{\partial y^a} y^a_i\right)$$

thus finishing the proof. □

Remark 4.5. The same result of proposition 4.3 applies to differentiable functions $f: J^1(C \times_M \mathcal{V}) \to \mathbb{R}$. In this case, the distribution is defined by

$$\mathcal{D}'_{(j^1_x\sigma, j^1_x\xi)} = \left\{\left(0_{j^1_x\sigma}, \left(X^{(1)}_{\mathcal{V}}\right)_{j^1_x\xi}\right) \middle| X \in \operatorname{gau} P, X_x = 0\right\} \tag{21}$$

and the function f must factor through

$$\bar{\mathcal{P}}: J^1(C \times_M \mathcal{V}) \to J^1C \times_M \pi^*_{\mathcal{V}}(\oplus^k T^*M) \qquad \bar{\mathcal{P}}\left(j^1_x\sigma, j^1_x\xi\right) = \left(j^1_x\sigma, \bar{\mathcal{P}}_{\mathcal{V}}\left(j^1_x\xi\right)\right). \tag{22}$$

Remark 4.6. Let L be a Lagrangian such that $L = \bar{L} \circ \bar{\mathcal{P}}$ on a dense open subset of $J^1(C \times_M \mathcal{V})$, as prescribed in proposition 4.3 and remark 4.5. Then L can be locally expressed as

$$L = \bar{L}\left(x^i, y^a, \frac{\partial \rho_s}{\partial y^b} y^b_i, A^\alpha_i, A^\alpha_{i,j}\right)$$

$\rho_1, \ldots, \rho_k$ being a system of generators of the algebra I^G.

Proposition 4.7. *If $L: J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ is a first integral of the distribution (21), then, for every $X \in \operatorname{gau} P$, the function $\bar{X}^{(1)}(L)$ is also a first integral of such a distribution.*

Proof. Let

$$\rho^s_i = \frac{\mathrm{d}\rho^s}{\mathrm{d}x^i} = \frac{\partial \rho^s}{\partial y^a} y^a_i$$

be the total derivative in $J^1(C \times_M \mathcal{V})$ of ρ^s with respect to the variable x^i. From remarks 4.5 and 4.6 and by using the chain rule, we have

$$\bar{X}^{(1)}L = \left(\frac{\partial \bar{L}}{\partial y^a} \circ \bar{\mathcal{P}}\right) \bar{X}^{(1)}(y^a) + \left(\frac{\partial \bar{L}}{\partial A^\alpha_i} \circ \bar{\mathcal{P}}\right) \bar{X}^{(1)}\left(A^\alpha_i\right) + \left(\frac{\partial \bar{L}}{\partial A^\alpha_{i,j}} \circ \bar{\mathcal{P}}\right) \bar{X}^{(1)}\left(A^\alpha_{i,j}\right)$$

as $\bar{X}^{(1)}\left(\rho^s_i\right) = 0$, as follows from proposition 4.1. Hence, as the formulae (6) and (7) show, $\bar{X}^{(1)}L$ factors through $\bar{\mathcal{P}}$. □

Corollary 4.8. *If a Lagrangian $L: J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ factors through $\bar{\mathcal{P}}$ on the dense open set as in remark 4.6, then its current form is gauge invariant.*

Proof. The current form of L is gauge invariant if and only if the formula (14) holds. From proposition 4.7 we know that $\bar{X}^{(1)}(L)$ is a first integral of the distribution (21). This allows us to conclude the proof. □

Proposition 4.9. *A Lagrangian $L: J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ is a first integral of the distribution on $J^1(C \times_M \mathcal{V})$ defined by*

$$\mathcal{D}''_{(j^1_x\sigma, j^1_x\xi)} = \left\{\bar{X}^{(1)}_{(j^1_x\sigma, j^1_x\xi)} \middle| X_x = 0\right\} \qquad \left(j^1_x\sigma, j^1_x\xi\right) \in J^1(C \times_M \mathcal{V})$$

if and only if L factors through the generalized curvature mapping (1).

Proof. The map $\Omega\colon J^1(C\times_M\mathcal{V})\to\mathcal{V}\oplus(T^*M\otimes\mathcal{V})\oplus(\bigwedge^2 T^*M\otimes\mathrm{ad}\,P)$ is a bundle projection with $\frac{1}{2}mn(n+3)$-dimensional fibres (see [1, 2]). From the local expression of the vector field $\bar{X}^{(1)}$ in equations (6) and (7) above, for $X\in\mathrm{gau}\,P$ with $X_x=0$, we obtain the following system of generators for the distribution $\mathcal{D}''$:

$$-\frac{\partial}{\partial A_j^\alpha}+c_{\alpha\gamma}^\beta A_i^\gamma\frac{\partial}{\partial A_{i,j}^\beta}-(\dot{B}_\alpha)_b^a y^b\frac{\partial}{\partial y_j^a}$$
$$\frac{\partial}{\partial A_{j,i}^\alpha}+\frac{\partial}{\partial A_{i,j}^\alpha}.$$

It is easily seen that these $\frac{1}{2}mn(n+3)$ vector fields are independent and Ω-vertical. Hence the integral leaves of the distribution $\mathcal{D}''$ are the fibres of Ω, thus finishing the proof. □

5. Lagrangians with gauge-invariant current form

Lemma 5.1. *Let Z_α, $\alpha=1,\ldots,s$, be vector fields on a manifold N linearly independent at every point in N, such that $[Z_\alpha,Z_\beta]=c_{\alpha\beta}^\gamma Z_\gamma$, for some constants $(c_{\alpha\beta}^\gamma)$, $\alpha,\beta,\gamma=1,\ldots,s$, and let $\Phi_1,\ldots,\Phi_s$ be smooth functions on N. Then, the system of equations*

$$Z_\alpha(L)=\Phi_\alpha\qquad 1\leqslant\alpha\leqslant s \tag{23}$$

admits locally a solution $L\in C^\infty(N)$ if and only if the following expressions hold:

$$Z_\alpha(\Phi_\beta)-Z_\beta(\Phi_\alpha)=c_{\alpha\beta}^\gamma\Phi_\gamma. \tag{24}$$

Proof. Let y be the natural coordinate in the real line $\mathbb{R}$. On $N\times\mathbb{R}$ we consider the s-dimensional distribution $\mathcal{T}$ spanned by the vector fields

$$Z'_\alpha=Z_\alpha+\Phi_\alpha\frac{\partial}{\partial y}.$$

By virtue of (24), we obtain

$$[Z'_\alpha,Z'_\beta]=c_{\alpha\beta}^\gamma Z'_\gamma.$$

Hence $\mathcal{T}$ is integrable. If $F\in C^\infty(N\times\mathbb{R})$ is a first integral of $\mathcal{T}$ transversal to $\partial/\partial y$, which exists by Frobenius theorem (as $n+1-s\geqslant 1$), then we can define a function $L\in C^\infty(N)$ by the equation

$$F(x,L(x))=0\qquad\forall x\in N$$

and it is readily checked that L satisfies (23). □

Theorem 5.2. *Let $P\to M$ be a principal G-bundle and let $\mathcal{V}\to M$ be the vector bundle associated with P by a linear representation of G on a finite-dimensional vector space V. Then a Lagrangian $L\colon J^1(C\times_M\mathcal{V})\to\mathbb{R}$ defines a gauge-invariant current form J_L if and only if L can be written as*

$$L=L'+L'' \tag{25}$$

where L' is a gauge-invariant Lagrangian and L'' is a Lagrangian factoring through $\bar{\mathcal{P}}$ (see (22)) on the dense open subset defined in remark 4.6.

Proof. We already know (see corollaries 3.2 and 4.8) that if L is as in formula (25) then its current form is gauge invariant.

Conversely, let L be a Lagrangian whose current form is gauge invariant. Then, the formula (14) can be rewritten as

$$Y_\alpha^i(\bar{X}^{(1)}(L)) = 0 \qquad \forall X \in \text{gau}\, P \tag{26}$$

where $\bar{X}^{(1)} = X_C^{(1)} + X_{\mathcal{V}}^{(1)}$, $X_C^{(1)}$, $X_{\mathcal{V}}^{(1)}$ are defined by the formulae (6), (7), and

$$Y_\alpha^i = (\dot{B}_\alpha)_b^a y^b \frac{\partial}{\partial y_i^a}. \tag{27}$$

As the values of $\partial^2 g^\alpha/\partial x^i \partial x^j$, $\partial g^\alpha/\partial x^i$ and g^α are pointwise arbitrary, we can study separately the parts in (26) with two, one or no derivatives of the functions g, thus obtaining the following groups of equations:

$$\begin{cases} (\dot{B}_\alpha)_b^a y^b \dfrac{\partial}{\partial y_i^a}\left(\dfrac{\partial L}{\partial A_{j,k}^\beta} + \dfrac{\partial L}{\partial A_{k,j}^\beta}\right) = 0 & \forall \alpha, \beta, i, j, k \\ (\dot{B}_\alpha)_b^a y^b \dfrac{\partial}{\partial y_i^a}\left(-\dfrac{\partial L}{\partial A_j^\beta} + c_{\beta\gamma}^\rho A_k^\gamma \dfrac{\partial L}{\partial A_{k,j}^\rho} - (\dot{B}_\beta)_b^a y^b \dfrac{\partial L}{\partial y_j^a}\right) = 0 & \forall \alpha, \beta, i, j \\ (\dot{B}_\alpha)_b^a y^b \dfrac{\partial}{\partial y_i^a}\left(c_{\beta\gamma}^\rho A_j^\gamma \dfrac{\partial L}{\partial A_j^\rho} + c_{\beta\gamma}^\rho A_{j,k}^\gamma \dfrac{\partial L}{\partial A_{j,k}^\rho}\right. & \\ \qquad \left. - (\dot{B}_\beta)_b^a y^b \dfrac{\partial L}{\partial y^a} - (\dot{B}_\beta)_b^a y_j^b \dfrac{\partial L}{\partial y_j^a}\right) = 0 & \forall \alpha, \beta, i \end{cases} \tag{28}$$

We now consider the first two groups of equations in (28). They define the following two integrable distributions: $\mathcal{D}'$, spanned by the vector fields $\{Y_\alpha^i\}$ in the formula (27) and $\mathcal{D}''$, spanned by

$$\begin{array}{ll} \dfrac{\partial}{\partial A_{j,k}^\alpha} + \dfrac{\partial}{\partial A_{k,j}^\alpha} & \forall \alpha, \forall j \leqslant k \\ -\dfrac{\partial}{\partial A_j^\beta} + c_{\beta\gamma}^\rho A_k^\gamma \dfrac{\partial}{\partial A_{k,j}^\rho} - (\dot{B}_\beta)_b^a y^b \dfrac{\partial}{\partial y_j^a} & \forall \alpha, \forall j \end{array}$$

The generators of $\mathcal{D}'$ commute with those of $\mathcal{D}''$. We can write the first two groups of equations in (28) as

$$\begin{cases} \dfrac{\partial L}{\partial A_{j,k}^\beta} + \dfrac{\partial L}{\partial A_{k,j}^\beta} = \Phi_\beta^{j,k} & \forall \beta, j, k \\ -\dfrac{\partial L}{\partial A_j^\beta} + c_{\beta\gamma}^\rho A_k^\gamma \dfrac{\partial L}{\partial A_{k,j}^\rho} - (\dot{B}_\beta)_b^a y^b \dfrac{\partial L}{\partial y_j^a} = \Phi_\beta^j & \forall \beta, j \end{cases} \tag{29}$$

where $\Phi_\beta^{j,k}$ and Φ_β^j are first integrals of $\mathcal{D}'$. Then, according to remarks 4.5 and 4.6, we have

$$\Phi_\beta^{j,k} = \Phi_\beta^{j,k}(x^i, y^a, \rho_i^s, A_i^\alpha, A_{i,j}^\alpha) \qquad \text{and} \qquad \Phi_\beta^j = \Phi_\beta^j(x^i, y^a, \rho_i^s, A_i^\alpha, A_{i,j}^\alpha)$$

where ρ^s, $s = 1, \ldots, r$, are generators of the algebra of G-invariant polynomials. The change of variables

$$A_i^\alpha \mapsto A_i^\alpha \qquad y^a \mapsto y^a \qquad y_i^a \mapsto \bar{y}_i^a = y_i^a - (\dot{B}_\beta)_d^a y^d A_i^\beta \tag{30}$$

which yields

$$\frac{\partial}{\partial A_i^\alpha} \mapsto \frac{\partial}{\partial A_i^\alpha} - (\dot{B}_\alpha)_d^a y^d \frac{\partial}{\partial \bar{y}_i^a}$$
$$\frac{\partial}{\partial y^a} \mapsto \frac{\partial}{\partial y^a} - (\dot{B}_\gamma)_a^d A_i^\gamma \frac{\partial}{\partial \bar{y}_i^d}$$
$$\frac{\partial}{\partial y_i^a} \mapsto \frac{\partial}{\partial \bar{y}_i^a}$$

transforms equations (29) into

$$\begin{aligned} &\frac{\partial L}{\partial A_{j,k}^\beta} + \frac{\partial L}{\partial A_{k,j}^\beta} = \Phi_\beta^{j,k} \qquad \forall \beta, j, k \\ &-\frac{\partial L}{\partial A_j^\beta} + c_{\beta\gamma}^\rho A_k^\gamma \frac{\partial L}{\partial A_{k,j}^\rho} = \Phi_\beta^j \qquad \forall \beta, j. \end{aligned} \tag{31}$$

The solution of this system is $L = L_1 + L_2$, L_1 being a particular solution and L_2 a solution of the associated homogeneous system; that is, a function factoring through the curvature mapping Ω (see proposition 4.9). We now look for a particular solution of the type $L_1 = L_1\left(x^i, y^a, \rho_i^s, A_i^\alpha, A_{i,j}^\alpha\right)$. Indeed, the system (31) satisfies the conditions of the lemma 5.1 on the domain of the coordinates $\left(x^i, A_i^\alpha, A_{i,j}^\alpha, y^a, \bar{y}_i^a\right)$, and hence, L being a solution, the compatibility conditions (24) hold true. Moreover, the vector fields defining the system (31) can be understood in the submanifold coordinated by $\left(x^i, A_i^\alpha, A_{i,j}^\alpha, y^a\right)$, where the same compatibility conditions (24) still hold. Hence, we can consider the variables ρ_i^s in $\Phi_\beta^{j,k}$, Φ_β^j as parameters instead of variables, and the existence of a solution L_1 depending on the variables $\left(x^i, A_i^\alpha, A_{i,j}^\alpha, y^a\right)$ and the parameters ρ_i^s is thus guaranteed. Therefore, we have

$$L = L_1\left(x^i, y^a, \rho_i^s, A_i^\alpha, A_{i,j}^\alpha\right) + L_2\left(x^i, y^a, \bar{y}_i^a, \Omega_{ij}^\alpha\right)$$

with $\Omega_{ij}^\alpha = A_{i,j}^\alpha - A_{j,i}^\alpha - c_{\beta\gamma}^\alpha A_i^\beta A_j^\gamma$. Since the Lagrangian L_1 already satisfies equations (25) (see corollary 4.8), we only need to impose the third group of equations in (28) onto the Lagrangian L_2. After the change of variables (30), these equations become

$$(\dot{B}_\alpha)_b^a y^b \frac{\partial}{\partial \bar{y}_i^a}\left(-(\check{B}_\beta)_\delta^\gamma \Omega_{kj}^\delta \frac{\partial L_2}{\partial \Omega_{kj}^\gamma} - (\dot{B}_\beta)_b^a y^b \frac{\partial L_2}{\partial y^a} - (\dot{B}_\beta)_b^a \bar{y}_j^b \frac{\partial L_2}{\partial \bar{y}_j^a}\right) = 0$$

where $B \mapsto (\check{B})_\delta^\gamma$ is the matrix expression with respect to the basis $\{B_i\}$ of the adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \simeq \mathfrak{gl}(m, \mathbb{R})$. If we put

$$Z_\beta = -(\check{B}_\beta)_\delta^\gamma \Omega_{kj}^\delta \frac{\partial}{\partial \Omega_{kj}^\gamma} - (\dot{B}_\beta)_b^a y^b \frac{\partial}{\partial y^a} - (\dot{B}_\beta)_b^a \bar{y}_j^b \frac{\partial}{\partial \bar{y}_j^a}$$

then the equations we have to solve become $Y_\alpha^i(Z_\beta(L_2)) = 0$, or equivalently,

$$Z_\beta(L_2) = \Phi_\beta \tag{32}$$

where the functions

$$\Phi_\beta = \Phi_\beta\left(x^i, y^a, \rho_i^s, \Omega_{ij}^\alpha\right) \tag{33}$$

are first integrals of the distribution $\mathcal{D}'$. The general solution of (32) is of the form $L_2 = L_0 + L_{\mathrm{gau}}$, where L_0 is a particular solution and L_{gau} is a solution of the homogeneous equation $Z_\beta(L_{\mathrm{gau}}) = 0$, that is, a gauge-invariant Lagrangian. We look for a particular solution of the type

$$L_0 = L_0\big(x^i, y^a, \rho_i^s, \Omega_{ij}^\alpha\big). \tag{34}$$

First we note that the functions ρ_i^s are first integrals of the vector fields Z_β. Indeed, we have

$$\begin{aligned} Z_\beta\big(\rho_i^s\big) &= -(\dot{B}_\beta)_b^a y^b \frac{\partial \rho_i^s}{\partial y^a} - (\dot{B}_\beta)_b^a \bar{y}_j^b \frac{\partial \rho_i^s}{\partial \bar{y}_j^a} \\ &= -(\dot{B}_\beta)_b^a y^b \bar{y}_i^d \frac{\partial^2 \rho^s}{\partial y^a \partial y^d} - (\dot{B}_\beta)_b^a \bar{y}_i^b \frac{\partial \rho_s}{\partial y^a} \\ &= -\bar{y}_i^d \frac{\partial}{\partial y^d}\left((\dot{B}_\beta)_b^a y^b \frac{\partial \rho^s}{\partial y^a}\right) \end{aligned}$$

which identically vanishes, as ρ^s is G-invariant.

Hence, if we now define

$$T_\beta = -(\check{B}_\beta)_\delta^\gamma \Omega_{kj}^\delta \frac{\partial}{\partial \Omega_{kj}^\gamma} - (\dot{B}_\beta)_b^a y^b \frac{\partial}{\partial y^a}$$

then we obtain $Z_\beta(L_0) = T_\beta(L_0)$, where the ρ_i^s of L_0 in $T_\beta(L_0)$ are considered as parameters. Similarly

$$Z_\beta(\Phi_\gamma) = T_\beta(\Phi_\gamma). \tag{35}$$

As $[Z_\beta, T_\gamma] = -c_{\beta\gamma}^\rho Z_\rho$ and $[T_\beta, T_\gamma] = -c_{\beta\gamma}^\rho T_\rho$, the systems of vector fields $Z_1, \ldots, Z_m$ and $T_1, \ldots, T_m$, both satisfy the conditions of lemma 5.1. According to this lemma, as the system (32) admits the solution L_2, we have

$$Z_\gamma(\Phi_\beta) - Z_\beta(\Phi_\gamma) = c_{\gamma\beta}^\rho \Phi_\rho.$$

But from (35) we can set $T_\gamma(\Phi_\beta) - T_\beta(\Phi_\gamma) = c_{\gamma\beta}^\rho \Phi_\rho$, where the ρ_i^s in Φ_α are considered as parameters. Hence, lemma 5.1 applied to the domain of the coordinates $\big(x^i, y^a, \Omega_{ij}^\alpha\big)$, guarantees the existence of a solution L_0 depending on these variables and on the parameters ρ_i^s, thus finishing the proof of the theorem. □

6. Some examples

6.1. The Abelian case

We consider $G = U(1)$ and its natural action on $V = \mathbb{C}$, i.e., $(\mathrm{e}^{\mathrm{i}\theta}, z) \mapsto \mathrm{e}^{\mathrm{i}\theta} z$, with $z = (y^1, y^2) \in V$, $\mathrm{e}^{\mathrm{i}\theta} \in U(1)$. For any principal $U(1)$-bundle $P \to M$, we also consider the associated bundle $\mathcal{V} \to M$. Actually, the usual setting for electromagnetic fields interacting with matter fields is $P = M \times U(1)$ and $M = \mathbb{R}^4$, or even a not necessarily trivial principal bundle P when one is dealing with monopoles. The action above (and all irreducible representations of $U(1)$) has the only generator $\rho(z) = \|z\|^2 = (y^1)^2 + (y^2)^2$ for the algebra of invariant polynomials. Hence the mappings $\bar{\mathcal{P}}_\mathcal{V}$ and $\bar{\mathcal{P}}$ have the following local expressions:

$$\begin{aligned} &\bar{\mathcal{P}}_\mathcal{V}\big(x^i, y^1, y^2, y_i^1, y_i^2\big) = \big(x^i, y^1, y^2, 2\big(y^1 y_i^1 + y^2 y_i^2\big)\big) \\ &\bar{\mathcal{P}}\big(x^i, A_i, A_{i,j}, y^1, y^2, y_i^1, y_i^2\big) = \big(x^i, A_i, A_{i,j}, y^1, y^2, 2\big(y^1 y_i^1 + y^2 y_i^2\big)\big). \end{aligned}$$

Therefore, theorem 4.4 claims that a Lagrangian $L\colon J^1\mathcal{V} \to \mathbb{R}$ is gauge invariant if and only if

$$L = \bar{L}\big(x^i, y^1, y^2, y^1 y_i^1 + y^2 y_i^2\big)$$

on an open dense subset O in $J^1\mathcal{V}$. Moreover, theorem 5.2 shows that a Lagrangian $L\colon J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ defines a gauge-invariant current form if and only if

$$L = L'\big(x^i, A_{i,j} - A_{j,i}, y^1, y^2, \bar{y}_i^1, \bar{y}_i^2\big) + L''\big(x^i, A_i, A_{i,j}, y^1, y^2, y^1 y_i^1 + y^2 y_i^2\big)$$

for L' and L'' are arbitrary functions.

6.2. The $SU(2)$ case

For the group $G = SU(2)$ acting on $V = \mathbb{R}^4 = \mathbb{C}^2$ in the natural way, i.e.,

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha z_1 + \beta z_2 \\ -\bar{\beta} z_1 + \bar{\alpha} z_2 \end{pmatrix}$$

for $a, \beta \in \mathbb{C}$ such that $\|\alpha\|^2 + \|\beta\|^2 = 1$, and $(z_1, z_2) \in \mathbb{C}^2$, with $z_1 = y^1 + \mathrm{i} y^2$, $z_2 = y^3 + \mathrm{i} y^4$, again we have a single generator of the algebra of invariant polynomials, which is $\rho(z_1, z_2) = \|z_1\|^2 + \|z_2\|^2$. Let $P \to M$ be a principal $SU(2)$-bundle and let $\mathcal{V} \to M$ be the associated bundle to P. This is the case considered for the models of weak interaction with fields. Hence, the mappings $\bar{\mathcal{P}}_{\mathcal{V}}$ and $\bar{\mathcal{P}}$ have the following local expressions:

$$\bar{\mathcal{P}}_{\mathcal{V}}\big(x^i, y^a, y_i^a\big) = \big(x^i, y^a, 2\big(y^1 y_i^1 + y^2 y_i^2 + y^3 y_i^3 + y^4 y_i^4\big)\big)$$

$$\bar{\mathcal{P}}\big(x^i, A_i^\alpha, A_{i,j}^\alpha, y^a, y_i^a\big) = \big(x^i, A_i^\alpha, A_{i,j}^\alpha, y^a, 2\big(y^1 y_i^1 + y^2 y_i^2 + y^3 y_i^3 + y^4 y_i^4\big)\big)$$

Therefore, a Lagrangian $L \colon J^1\mathcal{V} \to \mathbb{R}$ is gauge invariant if and only if

$$L = \bar{L}\big(x^i, y^a, y^1 y_i^1 + y^2 y_i^2 + y^3 y_i^3 + y^4 y_i^4\big)$$

on an open dense subset O in $J^1\mathcal{V}$. Moreover, theorem 5.2 shows that a Lagrangian $L \colon J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ defines a gauge-invariant current form if and only if, on an open dense subset, the following expression holds,

$$L = L'\big(x^i, \Omega_{ij}^\alpha, y^a, \bar{y}_i^a\big) + L''\big(x^i, A_i^\alpha, A_{i,j}^\alpha, y^a, y^1 y_i^1 + y^2 y_i^2 + y^3 y_i^3 + y^4 y_i^4\big)$$

for $\Omega_{ij}^\alpha = A_{i,j}^\alpha - A_{j,i}^\alpha - c_{\beta\gamma}^\alpha A_i^\beta A_j^\gamma$, where L' is invariant under the action of $SU(2)$ on $\mathcal{K}$ (see the formula (1) in the introduction) and L'' is arbitrary.

6.3. The spin case

Let FM be the principal bundle of oriented orthonormal frames on a Lorentzian manifold (M, h). Let us consider a spin structure; i.e., a principal-bundle morphism $P \to FM$ associated with the two-sheet covering $SL(2, \mathbb{C}) \to SO^0(1, 3)$ of the proper Lorentz group, where $P \to M$ is a principal $SL(2, \mathbb{C})$-bundle. We also consider the representation

$$\rho \colon SL(2, \mathbb{C}) \to GL(4, \mathbb{C})$$

$$\rho(A) = \begin{pmatrix} A & O \\ O & {}^t\bar{A}^{-1} \end{pmatrix}$$

and the associated vector bundle $\mathcal{V} = (P \times \mathbb{C}^4)/SL(2, \mathbb{C})$, which is used as a framework for the description of spinor fields; e.g., see [2, VI]. We now describe the basis of the algebra of invariant polynomials of the $SL(2, \mathbb{C})$-representation ρ given above. We have

$$\rho(A)(z, \zeta) = (A \cdot z, {}^t\bar{A}^{-1} \cdot \zeta) \qquad (z, \zeta) = (z_1, z_2, \zeta_1, \zeta_2) \in \mathbb{C}^4.$$

We claim that the Hermitian product $f \colon \mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ given by

$$f(z, \zeta) = \langle z, \zeta \rangle = \bar{z}^1 \zeta^1 + \bar{z}^2 \zeta^2$$

is invariant; precisely $f(\rho(A)(z, \zeta)) = f(z, \zeta)$. We put

$$z^1 = y^1 + \mathrm{i} y^2 \qquad z^2 = y^3 + \mathrm{i} y^4 \qquad \zeta^1 = y^5 + \mathrm{i} y^6 \qquad \zeta^2 = y^7 + \mathrm{i} y^8.$$

Hence, the functions

$$R = (\mathrm{Re}\, f)(z, \zeta) = y^1 y^5 + y^2 y^6 + y^3 y^7 + y^4 y^8$$
$$I = (\mathrm{Im}\, f)(z, \zeta) = y^1 y^6 - y^2 y^5 + y^3 y^8 - y^4 y^7$$

are $SL(2, \mathbb{C})$-invariants (cf [2, theorem 6.3.9]). It is easy to check that the $SL(2, \mathbb{C})$ acts freely on $\mathbb{C}^4 - X$, where $X = \{(z, \zeta) \in \mathbb{C}^4 : \langle z, \zeta \rangle = 0\}$; hence the dimension of the orbit of any $(z, \zeta) \notin X$ equals $\dim(SL(2, \mathbb{C})) = 6$. As $\dim \mathbb{C}^4 = 8$ and the two invariants above are functionally independent, any other invariant must depend on R and I. Hence, the mappings $\bar{\mathcal{P}}_{\mathcal{V}}$ and $\bar{\mathcal{P}}$ have the following local expressions,

$$\bar{\mathcal{P}}_{\mathcal{V}}(x^i, y^a, y^a_i) = (x^i, y^a, R_i, I_i)$$
$$\bar{\mathcal{P}}\big(x^i, A^\alpha_i, A^\alpha_{i,j}, y^a, y^a_i\big) = \big(x^i, A^\alpha_i, A^\alpha_{i,j}, y^a, R_i, I_i\big)$$

where

$$R_i = y^5 y^1_i + y^6 y^2_i + y^7 y^3_i + y^8 y^4_i + y^1 y^5_i + y^2 y^6_i + y^3 y^7_i + y^4 y^8_i$$
$$I_i = y^6 y^1_i - y^5 y^2_i + y^8 y^3_i - y^7 y^4_i - y^2 y^5_i + y^1 y^6_i - y^4 y^7_i + y^3 y^8_i.$$

Finally, a Lagrangian $L\colon J^1\mathcal{V} \to \mathbb{R}$ is gauge invariant if and only if on an open dense subset O in $J^1\mathcal{V}$ we have $L = \bar{L}(x^i, y^a, R_i, I_i)$. Similarly, a Lagrangian $L\colon J^1(C \times_M \mathcal{V}) \to \mathbb{R}$ defines a gauge-invariant current form if and only if, on an open dense subset L can be written as

$$L = L'\big(x^i, \Omega^\alpha_{ij}, y^a, \bar{y}^a_i\big) + L''\big(x^i, A^\alpha_i, A^\alpha_{i,j}, y^a, R_i, I_i\big)$$

for a function L' invariant under the action of $SL(2, \mathbb{C})$ on $\mathcal{K}$.

Remark 6.1. The group $SL(2, \mathbb{C})$ fails to be compact, and hence one could think that proposition 4.3 cannot be applied. Fortunately, it is known that every $SL(2, \mathbb{C})$-representation is isomorphic to a $SU(2)$-representation, that is, a representation of a compact group (see for example [15, corollary p 22]) and hence the results given above follow.

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